Homotopies of maps of suspended quaternionic projective spaces and their cohomotopy groups

Marek Golasiński^{1*}, Thiago de Melo², Edivaldo L. dos Santos³

¹ Faculty of Mathematics and Computer Science, University of Warmia and Mazury, Słoneczna 54 Street, 10-710 Olsztyn, Poland

² Instituto de Geociências e Ciências Exatas, UNESP-Univ Estadual Paulista, Av. 24A, 1515, Bela Vista, CEP 13.506-900, Rio Claro-SP, Brazil

³ Departamento de Matemática, Universidade Federal de São Carlos, Centro de Ciências Exatas e Tecnologia, CP 676, CEP 13565-905, São Carlos-SP, Brazil

E-mail: marekg@matman.uwm.edu.pl, thiago.melo@unesp.br, edivaldo@ufscar.br

Abstract

We describe the set [X, Y] of path-components of pointed mapping spaces $M_*(X, Y)$, where X is chosen to be the reduced k^{th} suspension $E^k \mathbb{H} P^m$ of a projective space $\mathbb{H} P^m$ for the skew algebra \mathbb{H} of quaternions and Y is a sphere \mathbb{S}^n or $\mathbb{F}P^n$ for \mathbb{F} being the real numbers \mathbb{R} , the complex numbers \mathbb{C} or \mathbb{H} . In particular, the cohomotopy sets $\pi^n(E^k \mathbb{H} P^m)$ are studied for $k \geq 0$ and certain $m, n \geq 1$.

2000 Mathematics Subject Classification. 54C35, 55P15. 58D15. Keywords, cofibration, cohomoty group, mapping space, path-component, projective space, coexact Puppe sequence.

Introduction

Let M(X,Y) be the mapping space of all continuous maps between connected spaces X and Y with the compact-open topology. These spaces are at the foundations of homotopy theory and appear in the literature dating back, at least, to Hurewicz's definition of the homotopy groups in the 1930s. Works focusing explicitly on the homotopy theory of a mapping space first appear in the 1940s. The space M(X,Y) is in general disconnected with path-components in one-to-one correspondence with the set $\langle X, Y \rangle$ of (free) homotopy classes of maps. Furthermore, different components may—and frequently do—have distinct homotopy types.

The space M(X,Y) has two close relatives. If X and Y are pointed spaces, we have $M_*(X,Y)$ the space of basepoint preserving maps with path-components in one-to-one correspondence with the set [X, Y] of based homotopy classes of maps. We write $M_f(X, Y)$ $(M_{f*}(X, Y))$ for the pathcomponent containing a given (based) map $f: X \to Y$.

A basic problem in homotopy theory is to determine whether two path-components are homotopy equivalent or, more generally, to classify the path-components of M(X,Y) (resp. $M_*(X,Y)$) up to homotopy type. Works on these classification problems date back to the 1940's. Whitehead [30, Theorem 2.8] considered the case $X = \mathbb{S}^m$, the *m*-sphere and proved that $M_f(\mathbb{S}^m, Y)$ is homotopy equivalent to $M_0(\mathbb{S}^m, Y)$ if and only if the evaluation fibration $\omega_f \colon M_f(\mathbb{S}^m, Y) \to Y$ admits a section, where 0 denotes the constant map. Then, results by Hansen [9], [10], and later by McClendon [20] treated this classification problem as well. The case in which X is a manifold and

Received by the editors: 15 January 2020. Accepted for publication: 14 June 2021.

^{*}The authors are supported by CAPES-Ciência sem Fronteiras, Processo: 88881.068125/2014-01

Y = BG, the classifying space of a compact Lie group G, has been the subject of extensive recent research by Crabb, Kono, Sutherland, Tsukuda, and others (see e.g., [3], [15], [27]). Then, Lupton and Smith [18] gave a general method that may be effectively applied to the question of whether two path-components of a mapping space M(X, Y) have the same homotopy type provided X is a co-H-space.

Now, let $\mathbb{F}P^n$ denote the projective *n*-space with $n \geq 1$ for \mathbb{F} being the real numbers \mathbb{R} , the complex numbers \mathbb{C} , or the skew \mathbb{R} -algebra \mathbb{H} of quaternions. In [5] and [6] the authors made use of Gottlieb groups of spheres to deal with path-components of the spaces $M(\mathbb{S}^m, \mathbb{S}^n)$ and $M(\mathbb{S}^m, \mathbb{F}P^n)$ for some $m, n \geq 1$, respectively. Then, [7] concerns to the set [X, Y] of path-components of mapping spaces $M_*(X, Y)$, where X is chosen to be $E^k \mathbb{F}P^m$ and Y is a sphere \mathbb{S}^n or $\mathbb{F}P^n$ for $\mathbb{F} = \mathbb{R}$ or \mathbb{C} only, where E^k stands for the reduced k^{th} suspension functor.

The purpose of this paper is to extend results of [7] and study the set [X, Y] of path-components of mapping spaces $M_*(X, Y)$, where X is chosen to be $E^k \mathbb{H}P^m$ and Y is a sphere \mathbb{S}^n or $\mathbb{F}P^n$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} . In particular, we describe cohomotopy groups $\pi^n(E^k \mathbb{H}P^m)$ for some $k \ge 0$ and certain $m, n \ge 1$.

The set $[\mathbb{H}P^n, \mathbb{H}P^n]$ of homotopy classes of pointed maps for $n \ge 2$ was extensively studied in [8, Section 3] and path-components of $M(\mathbb{H}P^n, \mathbb{H}P^n)$ were described for n = 2, 3.

Although a number of papers have been devoted to the theoretical aspects of cohomotopy, few cohomotopy groups have been computed. In [29] some cohomotopy groups of projective spaces $\pi^m(\mathbb{F}P^n)$ have been calculated by making use of the Puppe sequence. By [24], S-duality theorem reduces the problem for the projective spaces to the calculation of the homotopy groups of the Stiefel manifolds or the stunted quasi-projective spaces.

We follow [29] to update and extend computations of $\pi^m(\mathbb{H}P^n)$ on $\pi^{4n-k}(E^l\mathbb{H}P^n)$ for $k+l \leq 6$ with $k, l \geq 0$ and $n \geq 3$.

Section 1 fixes up some notations and definitions, and necessary results as well. Then, Section 2 presents homotopies of maps in $M(E^k \mathbb{F}P^m, \mathbb{H}P^n)$ and $M(E^k \mathbb{H}P^m, \mathbb{F}P^n)$ for $k \ge 0$ and certain $m, n \ge 1$. Proposition 2.3 shows that the degree of any non-trivial self-map on $\mathbb{H}P^2$ is non zero and there is an essential self-map on $\mathbb{H}P^3$ with the trivial degree. Such a result has been obtained by Marcum and Randall in [19] as well. A sufficient condition for the conjecture from [19] "... the space $\mathbb{H}P^n$ admits an essential self-map with trivial degree provided $n \ge 3$ " is stated in Remark 2.2. Then, Proposition 2.8 updates and extends results [29, (14.1)–(14.3) Theorems and Remark] on $\pi^{4n-k}(E^l \mathbb{H}P^n)$ for $k+l \le 6$ with $k \ge 0, l \ge 1$ and $n \ge 3$.

Crabb and Sutherland [2] (see also [21] and [23]) gave an explicit homotopy classification of path-components of $M(S, \mathbb{C}P^n)$, where S is any closed connected surface, and of $M(\mathbb{C}P^m, \mathbb{C}P^n)$, $M(\mathbb{C}P^m, \mathbb{R}P^n)$ and $M(\mathbb{R}P^m, \mathbb{R}P^n)$ for certain values of $m, n \geq 1$. When the domain is $\mathbb{R}P^m$, by [2, Propositions 2.3 and 2.4] there is an interesting connection of path-components with Hurwitz-Radon numbers.

We plan to use our presented results to extend Crabb and Sutherland's classifications of pathcomponents of $M(\mathbb{F}P^m, \mathbb{S}^n)$ and $M(\mathbb{F}P^m, \mathbb{F}P^n)$ for other $m, n \geq 1$ in a forthcoming paper.

1 Prerequisites

For topological spaces X and Y, let M(X, Y) be the space of all continuous maps equipped with the compact-open topology. In the pointed case, for this space we write $M_*(X, Y)$. Let $M_f(X, Y)$ (resp. $M_{*f}(X, Y)$) be the path-component of M(X, Y) (resp. $M_*(X, Y)$) containing a (resp. pointed) map $f: X \to Y$.

For pointed spaces X and Y, denote by $\pi_n(X)$ the n^{th} homotopy group of X and write $\langle X, Y \rangle$ and [X, Y] for the sets of homotopy classes of free and pointed maps, respectively. It is well known that there is an action of the fundamental group $\pi_1(Y)$ on [X, Y] and there is a bijection $\langle X, Y \rangle \approx [X, Y]/\pi_1(Y)$ provided X and Y are path-connected.

As it has been pointed out by Whitehead [30] all path-components of $M_*(\mathbb{S}^n, X)$ for the *n*-sphere \mathbb{S}^n have the same homotopy type. Moreover, Lang [16, Lemma 2.1] generalized this result for the space $M_*(EX, Y)$, where EX is the reduced suspension of the pointed space X. In general, distinct path-components of the space $M_*(X, Y)$ need not be homotopy equivalent.

We study the sets $[E^k \mathbb{F}P^m, \mathbb{F}P^n]$ and $[E^k \mathbb{F}P^m, \mathbb{S}^n]$ to classify path-components of the spaces $M_*(\mathbb{F}P^m, \mathbb{F}P^n)$ and $M_*(\mathbb{F}P^m, \mathbb{S}^n)$ up to homotopy type in a forthcoming paper.

Throughout the rest of this paper, all spaces are assumed to be path-connected, pointed compactly generated and all maps are pointed maps. Further, we do not distinguish between a map and its homotopy class and we freely use notations from Toda's book [28].

Recall that given a map $f: X \to Y$ there is a cofibration

$$X \xrightarrow{f} Y \hookrightarrow C_f,$$

where C_f is the mapping cone of f which yields the coexact Puppe sequence

$$X \xrightarrow{f} Y \hookrightarrow C_f \xrightarrow{\partial} EX \xrightarrow{-Ef} EY \to EC_f \to \dots \to E^k X \xrightarrow{(-1)^k E^k f} E^k Y \to E^k C_f \to \dots$$

where $\partial: C_f \to EX$ is the connecting map.

Next, for a CW-complex X with dim $X \leq 2n-2$, a group structure can be defined on $\pi^n(X) = [X, \mathbb{S}^n]$ in the following way. For $\alpha, \beta \in \pi^n(X)$, one considers the map

$$(\alpha \times \beta) \Delta \colon X \to \mathbb{S}^n \times \mathbb{S}^n,$$

where Δ is the diagonal map. In view of the restriction on the dimension of X, there is a unique homotopy class of maps $\gamma \colon X \to \mathbb{S}^n \vee \mathbb{S}^n$ such that compositions with the natural inclusion $\mathbb{S}^n \vee \mathbb{S}^n \hookrightarrow$ $\mathbb{S}^n \times \mathbb{S}^n$ are homotopic to $(\alpha \times \beta)\Delta$. The homotopy class $[\nabla \gamma] \in \pi^n(X)$, where $\nabla \colon \mathbb{S}^n \vee \mathbb{S}^n \to \mathbb{S}^n$ is the folding map, is set to $[\alpha] + [\beta] \in \pi^n(X)$. With respect to this operation the set $\pi^n(X)$ is an abelian group, called the n^{th} cohomotopy group of X. For a CW-complex X with dim X < n we have $\pi^n(X) = 0$. Thus, the functor π^n is of interest in dimensions from n to 2n - 2, that is, in the so-called stable range.

Proposition 1.1 ([14, Proposition 3.2.2]). If X is an *n*-connected CW-complex then the canonical map $X \to \Omega EX$ is a (2n + 1)-equivalence, for the loop functor Ω .

Proposition 1.1 leads to the following version of the Freudenthal Suspension Theorem (see [14, Corollary 3.2.3]).

Theorem 1.2. If Y is an *n*-connected CW-complex and X is a CW-complex then the maps of homotopy classes

$$[X,Y] \rightarrow [EX,EY] \rightarrow [E^2X,E^2Y] \rightarrow \cdots$$

are surjections if dim $X \leq 2n + 1$ and bijections if dim $X \leq 2n$. In particular, the induced group structure on [X, Y] is abelian.

In addition, if dim $X \leq 2n-2$ then $\pi^n(X) \approx \pi^{n+1}(EX)$. This isomorphism is given by the suspension map $E_* \colon [X, \mathbb{S}^n] \to [EX, E\mathbb{S}^n] \approx [EX, \mathbb{S}^{n+1}]$.

The n^{th} stable cohomotopy group of a space X is defined by

$$\pi_S^n(X) = \operatorname{colim}_k \pi^{n+k}(E^k X).$$

Notice that Theorem 1.2 implies an isomorphism

$$\pi^n_S(X) \approx \pi^{n+k}(E^k X)$$

provided $k \ge \dim X - (2n - 2)$.

Let us point out that the groups $\pi_S^n(\mathbb{F}P^m)$ have been described in [24] by means of homotopy groups of Stiefel manifolds. Other results on cohomotopy groups can be found in [26].

Let \mathbb{F} be the field of real numbers \mathbb{R} , complex numbers \mathbb{C} or the skew quaternion algebra \mathbb{H} . Write $\gamma_{n,\mathbb{F}} \colon \mathbb{S}^{d(n+1)-1} \to \mathbb{F}P^n$ for the canonical quotient map, $p_{n,\mathbb{F}} \colon \mathbb{F}P^n \to \mathbb{F}P^n / \mathbb{F}P^{n-1} = \mathbb{S}^{dn}$ for the pinching map, $i_{n,\mathbb{F}} \colon \mathbb{S}^d = \mathbb{F}P^1 \hookrightarrow \mathbb{F}P^n$ and $i_{m,n,\mathbb{F}} \colon \mathbb{F}P^m \hookrightarrow \mathbb{F}P^n$ for the canonical inclusion maps, where $d = \dim_{\mathbb{R}} \mathbb{F}$ is the dimension of \mathbb{F} over \mathbb{R} . Then, we recall the following results useful in the sequel.

Proposition 1.3. If $n \ge 1$ then:

(1, [11, (2.10)(a)] and [29, (5.1) Lemma and (9.2) Lemma]) The composite map

$$p_{n,\mathbb{F}}\gamma_{n,\mathbb{F}} = \begin{cases} (1+(-1)^{n-1})\iota_n & \text{if } \mathbb{F} = \mathbb{R};\\ n\eta_{2n} & \text{if } \mathbb{F} = \mathbb{C};\\ n\nu_{4n}^+ & \text{if } \mathbb{F} = \mathbb{H} \end{cases}$$

with $\nu_n^+ = \nu_n + \alpha_1(n)$ for $n \ge 4$.

(2, [1, Corollary 5.4.5]) $\overline{\Omega}(\mathbb{F}P^n) \simeq \Omega(\mathbb{S}^{d(n+1)-1}) \times \mathbb{S}^{d-1}.$

Notice that by dimension argument, the exact sequence

$$\cdots \to [E^{l+1} \mathbb{F}P^{n-m}, \mathbb{S}^{dn-k}] \to [E^l(\mathbb{F}P^n/\mathbb{F}P^{n-m}), \mathbb{S}^{dn-k}] \to \\ [E^l \mathbb{F}P^n, \mathbb{S}^{dn-k}] \to [E^l \mathbb{F}P^{n-m}, \mathbb{S}^{dn-k}] \to \cdots$$

determined by the Puppe sequence associated to the cofibration $\mathbb{F}P^{n-m} \hookrightarrow \mathbb{F}P^n \to \mathbb{F}P^n / \mathbb{F}P^{n-m}$ yields:

Proposition 1.4. If $k, l \ge 0$ and $1 \le m \le n-1$ then

$$[E^{l}\mathbb{F}P^{n},\mathbb{S}^{dn-k}]\approx[E^{l}(\mathbb{F}P^{n}/\mathbb{F}P^{n-m}),\mathbb{S}^{dn-k}]$$

provided $k + l \leq dm - 2$, where $d = \dim_{\mathbb{R}} \mathbb{F}$ and

$$[E^{l}\mathbb{F}P^{n},\mathbb{S}^{dn-k}] = \operatorname{coker}\left([E^{l+1}\mathbb{F}P^{n-m},\mathbb{S}^{dn-k}] \to [E^{l}(\mathbb{F}P^{n}/\mathbb{F}P^{n-m}),\mathbb{S}^{dn-k}]\right)$$

for k + l = dm - 1.

Consequently, Theorem 1.2 leads to

$$[\mathbb{F}P^n, \mathbb{S}^{dn-k-l}] \approx [\mathbb{F}P^n/\mathbb{F}P^{n-m}, \mathbb{S}^{dn-k-l}]$$

for $k, l \ge 0$ and $1 \le m \le n-1$ with $k+l \le dm-2$ provided $2(k+l+1) \le dn$, where $d = \dim_{\mathbb{R}} \mathbb{F}$. In the sequel, we also need [7, Lemma 3.4]:

Lemma 1.5. Let $f: X \to \mathbb{C}P^n$, for $n \ge 1$. Then, there is a map $\tilde{f}: X \to \mathbb{S}^{2n+1}$ with $f = \gamma_{n,\mathbb{C}}\tilde{f}$ if and only if the induced map $f^*: H^2(\mathbb{C}P^n; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ is trivial. Equivalently, the image of the induced map $(\gamma_{n,\mathbb{C}})_*: [X, \mathbb{S}^{2n+1}] \to [X, \mathbb{C}P^n]$ is given by maps $f: X \to \mathbb{C}P^n$ such that $f^*: H^2(\mathbb{C}P^n; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ is trivial.

In particular, the induced map $(\gamma_{n,\mathbb{C}})_* \colon [X, \mathbb{S}^{2n+1}] \to [X, \mathbb{C}P^n]$ is a bijection provided $[X, \mathbb{S}^1] = H^1(X; \mathbb{Z}) = H^2(X; \mathbb{Z}) = 0.$

2 Homotopies of $M(E^k \mathbb{F}P^m, \mathbb{H}P^n)$ and $M(E^k \mathbb{H}P^m, \mathbb{F}P^n)$

If X is a CW-complex then, in view of Proposition 1.3(2), there is an isomorphism

$$[EX, \mathbb{H}P^n] \approx [X, \mathbb{S}^3] \oplus [EX, \mathbb{S}^{4n+3}].$$

In particular, if dim $X \leq 4n + 1$ then there is an isomorphism

$$[EX, \mathbb{H}P^n] \xrightarrow{\approx} [X, \mathbb{S}^3]$$

Further, for a CW-complex X with dim $X \leq 4n + 2$ we have

$$[X, \mathbb{H}P^n] \approx [X, \mathbb{H}P^{n+1}] \approx [X, \mathbb{H}P^{\infty}].$$

This implies

$$[\mathbb{F}P^m, \mathbb{H}P^n] \approx \begin{cases} [\mathbb{R}P^m, \mathbb{H}P^\infty] & \text{if } \mathbb{F} = \mathbb{R} \text{ and } 1 \le m \le 4n+2; \\ [\mathbb{C}P^m, \mathbb{H}P^\infty] & \text{if } \mathbb{F} = \mathbb{C} \text{ and } 1 \le m \le 2n+1; \\ [\mathbb{H}P^m, \mathbb{H}P^\infty] & \text{if } \mathbb{F} = \mathbb{H} \text{ and } 1 \le m \le n. \end{cases}$$

Hence, the classification of line bundles over $\mathbb{H}P^n$ is the same as the classification of self-maps on $\mathbb{H}P^n$. Further, there is a chain of restriction maps

$$\mathbb{Z} \approx [\mathbb{H}P^1, \mathbb{H}P^1] \leftarrow [\mathbb{H}P^2, \mathbb{H}P^2] \leftarrow \dots \leftarrow [\mathbb{H}P^\infty, \mathbb{H}P^\infty].$$

Unfortunately, the classification of self-maps on $\mathbb{H}P^n$ for $n < \infty$ is still an open problem. Given f a self-map on $\mathbb{H}P^n$, the *degree* of f is the integer $\deg(f)$ such that $f^*(x) = \deg(f)x$ for the induced map $f^* \colon H^4(\mathbb{H}P^n;\mathbb{Z}) \to H^4(\mathbb{H}P^n;\mathbb{Z})$ and a generator $x \in H^4(\mathbb{H}P^n;\mathbb{Z}) \approx \mathbb{Z}$. Since the space $\mathbb{H}P^n$ is 3-connected, the degree of f might be defined equivalently as the homotopy class in $\pi_4(\mathbb{H}P^n) \approx \pi_3(\mathbb{S}^3) \approx \mathbb{Z}$ of the restriction of f to $\mathbb{S}^4 = \mathbb{H}P^1 \subseteq \mathbb{H}P^n$.

Hence, the effect of a self-map on homology is determined by its degree. If two self-maps on $\mathbb{H}P^n$ induce the same homomorphism in homology does it follow that they are homotopic? Marcum and Randall gave a negative answer to this question in [19]. For n = 3, 4, 5 they found an essential self-map on $\mathbb{H}P^n$ which induces the trivial homomorphism in homology.

By [21, Theorem 2.5], two self-maps on $\mathbb{H}P^n$ induce the same homomorphism in homology if and only if they are stably homotopic for $n \leq \infty$.

But, for $n = \infty$ we have the following situation. We say that a self-map $f: \mathbb{H}P^{\infty} \to \mathbb{H}P^{\infty}$ has degree deg(f) = d if the corresponding self-map on $\Omega(\mathbb{H}P^{\infty}) = \mathbb{S}^3$ has degree d in the usual sense. It is well known that deg(f) is zero or an odd square integer [4]. In view of [22, Classification Theorem], self-maps on $\mathbb{H}P^{\infty}$ are classified, up to homotopy, by their degree.

The cofibration $\mathbb{S}^{4n+3} \xrightarrow{\gamma_{n,\mathbb{H}}} \mathbb{H}P^n \hookrightarrow \mathbb{H}P^{n+1}$ shows that the obstruction to extending a map $f: \mathbb{H}P^n \to \mathbb{H}P^\infty$ to $\mathbb{H}P^{n+1}$ is $o(f) = f\gamma_{n,\mathbb{H}} \in \pi_{4n+3}(\mathbb{H}P^\infty)$. But, $f = i_{n,\infty,\mathbb{H}}f'$ for the canonical inclusion $i_{n,\infty,\mathbb{H}}: \mathbb{H}P^n \hookrightarrow \mathbb{H}P^\infty$ and some $f': \mathbb{H}P^n \to \mathbb{H}P^n$. Hence, alternatively we have $f\gamma_{n,\mathbb{H}} = i_{n,\infty,\mathbb{H}}f'\gamma_{n,\mathbb{H}}$.

Remark 2.1. Let $f: \mathbb{H}P^{n+1} \to \mathbb{H}P^n$. Since the map $i_{n,n+1,\mathbb{H}}\gamma_{n,\mathbb{H}}: \mathbb{S}^{4n+3} \to \mathbb{H}P^{n+1}$ is homotopy trivial, we get that the map $f_{i_{n,n+1,\mathbb{H}}}\gamma_{n,\mathbb{H}}$ is homotopy trivial as well. Thus, by quaternionic versions of [7, Lemma 3.1 and Remark 3.2], we have the relation $[f_{i_{n,n+1,\mathbb{H}}}\gamma_{n,\mathbb{H}}] = (\deg(f_{i_{n,n+1,\mathbb{H}}}))^{n+1}[\gamma_{n,\mathbb{H}}] = 0$ for homotopy classes in the group $\pi_{4n+3}(\mathbb{H}P^n)$.

Hence, $\deg(f_{i_{n,n+1,\mathbb{H}}}) = 0$ and consequently the image of the map

$$(i_{n,n+k,\mathbb{H}})^* \colon [\mathbb{H}P^{n+k},\mathbb{H}P^n] \to [\mathbb{H}P^n,\mathbb{H}P^n]$$

coincides with maps of trivial degree for $k \geq 1$.

Certainly, $[\mathbb{H}P^1, \mathbb{H}P^1] \approx \mathbb{Z}$. Further, the restriction map $[\mathbb{H}P^{n+1}, \mathbb{H}P^{n+1}] \rightarrow [\mathbb{H}P^n, \mathbb{H}P^n]$ for $n \geq 1$ might be included into an exact sequence. Namely, the Puppe sequence associated to the cofibration $\mathbb{S}^{4n+3} \xrightarrow{\gamma_{n,\mathbb{H}}} \mathbb{H}P^n \hookrightarrow \mathbb{H}P^{n+1}$ leads to the exact sequence

$$\cdots \rightarrow [E^2 \mathbb{H}P^{n+1}, \mathbb{H}P^{n+1}] \rightarrow [E^2 \mathbb{H}P^n, \mathbb{H}P^{n+1}] \rightarrow [\mathbb{S}^{4n+5}, \mathbb{H}P^{n+1}] \rightarrow [E \mathbb{H}P^{n+1}, \mathbb{H}P^{n+1}] \rightarrow [E \mathbb{H}P^n, \mathbb{H}P^{n+1}] \rightarrow [\mathbb{S}^{4n+4}, \mathbb{H}P^{n+1}] \rightarrow [\mathbb{H}P^{n+1}, \mathbb{H}P^{n+1}] \rightarrow [\mathbb{H}P^n, \mathbb{H}P^{n+1}] \rightarrow [\mathbb{S}^{4n+3}, \mathbb{H}P^{n+1}],$$

or equivalently (using skeleton arguments and the homotopy equivalence $\mathbb{S}^3 \simeq \Omega(\mathbb{H}P^{\infty})$), to the following one

$$\cdots \to [E \mathbb{H} P^{n+1}, \mathbb{S}^3] \to [E \mathbb{H} P^n, \mathbb{S}^3] \to [\mathbb{S}^{4n+4}, \mathbb{S}^3] \to [\mathbb{H} P^{n+1}, \mathbb{S}^3]$$

$$\to [\mathbb{H} P^n, \mathbb{S}^3] \xrightarrow{\partial^*} [\mathbb{S}^{4n+3}, \mathbb{S}^3] \to [\mathbb{H} P^{n+1}, \mathbb{H} P^{n+1}] \to [\mathbb{H} P^n, \mathbb{H} P^n] \to [\mathbb{S}^{4n+2}, \mathbb{S}^3].$$

$$(1.1)$$

Remark 2.2. Recall that, by [19, Conjecture] the space $\mathbb{H}P^{n+1}$ admits an essential self-map with trivial degree provided $n \geq 2$. This was verified for n = 2, 3, 4 in [19] and then for n = 5, 6 in [25]. Notice that this Conjecture will follow from (1.1) if we can show that the map $\partial^* : [\mathbb{H}P^n, \mathbb{S}^3] \to [\mathbb{S}^{4n+3}, \mathbb{S}^3]$ is not surjective.

Proof. To prove the last statement, let the map $\partial^* \colon [\mathbb{H}P^n, \mathbb{S}^3] \to [\mathbb{S}^{4n+3}, \mathbb{S}^3]$ be not surjective. Then, by the exact sequence

$$0 \to \operatorname{Im} \partial^* \to [\mathbb{S}^{4n+3}, \mathbb{S}^3] \to [\mathbb{H}P^{n+1}, \mathbb{H}P^{n+1}] \to [\mathbb{H}P^n, \mathbb{H}P^n] \to [\mathbb{S}^{4n+2}, \mathbb{S}^3],$$

the map $[\mathbb{S}^{4n+3}, \mathbb{S}^3] \to [\mathbb{H}P^{n+1}, \mathbb{H}P^{n+1}]$ is non trivial.

Now, for $\alpha \in [\mathbb{S}^{4n+3}, \mathbb{S}^3] \setminus \operatorname{Im} \partial^*$, we take its adjoint $\bar{\alpha} \colon \mathbb{S}^{4n+4} \to \mathbb{H}P^{\infty}$. Thus, the map $\bar{\alpha}p_{n+1,\mathbb{H}} \colon \mathbb{H}P^{n+1} \to \mathbb{H}P^{\infty}$ determines an essential self-map $f \colon \mathbb{H}P^{n+1} \to \mathbb{H}P^{n+1}$ with $\operatorname{deg}(f) = 0$ and the proof is complete.

Now, we analyse the restriction maps $[\mathbb{H}P^2, \mathbb{H}P^2] \rightarrow [\mathbb{H}P^1, \mathbb{H}P^1]$ and $[\mathbb{H}P^3, \mathbb{H}P^3] \rightarrow [\mathbb{H}P^2, \mathbb{H}P^2]$ to state:

Proposition 2.3. There are exact sequences of pointed sets:

- (1) $0 \to [\mathbb{H}P^2, \mathbb{H}P^2] \to [\mathbb{H}P^1, \mathbb{H}P^1] \to [\mathbb{S}^6, \mathbb{S}^3]$. This implies that only the trivial self-map on $\mathbb{H}P^2$ has the trivial degree.
- (2) $0 \to [\mathbb{S}^{11}, \mathbb{S}^3] \to [\mathbb{H}P^3, \mathbb{H}P^3] \to [\mathbb{H}P^2, \mathbb{H}P^2] \to [\mathbb{S}^{10}, \mathbb{S}^3]$. This implies the existence of an essential self-map on $\mathbb{H}P^3$ with the trivial degree.

We point out that (1) and (2) have been already shown, using different methods, by Marcum and Randall in [19].

Proof. (1): First, to make use of the exact sequence (1.1), we show that the map $[\mathbb{S}^4, \mathbb{S}^3] = [\mathbb{H}P^1, \mathbb{S}^3] \to [\mathbb{S}^7, \mathbb{S}^3]$ is surjective. In view of [28, Chapter V], we have $[\mathbb{H}P^1, \mathbb{S}^3] = \pi_4(\mathbb{S}^3) = \mathbb{Z}_2\{\eta_3\}, [\mathbb{S}^7, \mathbb{S}^3] = \pi_7(\mathbb{S}^3) = \mathbb{Z}_2\{\nu'\eta_6\}$ and, by [28, (5.9)], $\eta_3\nu_4 = \nu'\eta_6$. But, as it is well-known, $\gamma_{1,\mathbb{H}} \equiv \pm(\nu_4 + \alpha_1(4)) \pmod{E\nu'}, \eta_3\alpha_1(4) = 0$ and, by [28, Lemma 5.7], it holds $E(\eta_2\nu') = 0$. Hence, the map $[\mathbb{S}^4, \mathbb{S}^3] = [\mathbb{H}P^1, \mathbb{S}^3] \to [\mathbb{S}^7, \mathbb{S}^3]$ is surjective and, by means of (1.1), we get the exact sequence

$$0 \to [\mathbb{H}P^2, \mathbb{H}P^2] \to [\mathbb{H}P^1, \mathbb{H}P^1] \to [\mathbb{S}^6, \mathbb{S}^3]$$

and so only the trivial self-map on $\mathbb{H}P^2$ restricts to the trivial self-map on $\mathbb{H}P^1$.

(2): To analyse $[\mathbb{H}P^3, \mathbb{H}P^3]$, we first compute $[\mathbb{H}P^2, \mathbb{S}^3]$. For this purpose, consider the exact sequence

$$\cdots \to [E\mathbb{H}P^1, \mathbb{S}^3] \to [\mathbb{S}^8, \mathbb{S}^3] \to [\mathbb{H}P^2, \mathbb{S}^3] \to [\mathbb{H}P^1, \mathbb{S}^3] \to [\mathbb{S}^7, \mathbb{S}^3]$$

determined by the Puppe sequence associated to the cofibration $\mathbb{S}^7 \to \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^2$. Since, by the proof of (1), the map $[\mathbb{S}^4, \mathbb{S}^3] = [\mathbb{H}P^1, \mathbb{S}^3] \to [\mathbb{S}^7, \mathbb{S}^3]$ is also an isomorphism, we derive that the map $[\mathbb{H}P^2, \mathbb{S}^3] \to [\mathbb{H}P^1, \mathbb{S}^3]$ is trivial.

Now, we show that the map $[E\mathbb{H}P^1, \mathbb{S}^3] \to [\mathbb{S}^8, \mathbb{S}^3]$ is trivial as well. But, $[E\mathbb{H}P^1, \mathbb{S}^3] = \pi_5(\mathbb{S}^3) = \mathbb{Z}_2\{\eta_4\eta_3\}$ and $E(\eta_2\nu') = 0$. Even though $\eta_3\nu_4 = \nu'\eta_6 \neq 0$, by $E\gamma_{1,\mathbb{H}} \equiv \pm(\nu_5 + \alpha_1(5)) \pmod{E^2\nu'}$ and $E^2\nu' = 2\nu_5 ([28, (5.5)])$, we get $\eta_3\eta_4 E\gamma_{1,\mathbb{H}} = \eta_3\eta_4\nu_5 = \eta_3 E(\eta_3\nu_4) = \eta_3 E(\nu'\eta_6) = E(\eta_2\nu')\eta_7 = 0$. Finally, we have a bijection

$$\mathbb{Z}_{2}\{\nu'\eta_{6}\eta_{7}\} = [\mathbb{S}^{8}, \mathbb{S}^{3}] \xrightarrow[\approx]{(p_{2,\mathbb{H}})^{*}} [\mathbb{H}P^{2}, \mathbb{S}^{3}].$$

Furthermore, $[\mathbb{S}^8, \mathbb{S}^3] = \pi_8(\mathbb{S}^3) = \mathbb{Z}_2\{\nu'\eta_6\eta_7\}$ and, by Proposition 1.3(1), we have the relation $\nu'\eta_6\eta_7p_{2,\mathbb{H}}\gamma_{2,\mathbb{H}} = \nu'\eta_6\eta_7(2\nu_8) = 0$. Hence, we get that the map $\partial^* : [\mathbb{H}P^2, \mathbb{S}^3] \to [\mathbb{S}^{11}, \mathbb{S}^3]$ is trivial. Consequently, in view of (1.1), we obtain the exact sequence

$$0 \to [\mathbb{S}^{11}, \mathbb{S}^3] \to [\mathbb{H}P^3, \mathbb{H}P^3] \to [\mathbb{H}P^2, \mathbb{H}P^2] \to [\mathbb{S}^{10}, \mathbb{S}^3],$$

where $\mathbb{Z}_2\{\varepsilon_3\} = [\mathbb{S}^{11}, \mathbb{S}^3] \approx [\mathbb{S}^{12}, \mathbb{H}P^{\infty}] = \mathbb{Z}_2\{\bar{\varepsilon}_3\}$ for $\bar{\varepsilon}_3 \colon \mathbb{S}^{12} \to \mathbb{H}P^{\infty}$ being the adjoint of $\varepsilon_3 \colon \mathbb{S}^{11} \to \mathbb{S}^3$. Then, the map $\bar{\varepsilon}_3 p_{3,\mathbb{H}} \colon \mathbb{H}P^3 \to \mathbb{H}P^{\infty}$ determines an essential self-map $f \colon \mathbb{H}P^3 \to \mathbb{H}P^3$ with $\deg(f) = 0$ and the proof is complete.

Proposition 2.3 and Remark 2.1 yield:

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Corollary 2.4. There is a bijection

$$\pi_{11}(\mathbb{S}^3) = \mathbb{Z}_2\{\varepsilon_3\} \approx [\mathbb{S}^{12}, \mathbb{H}P^2] \xrightarrow[\approx]{(p_{3,\mathbb{H}})^*} [\mathbb{H}P^3, \mathbb{H}P^2].$$

Proof. Proposition 2.3(1) and Remark 2.1 imply that the map

$$(i_{2,3,\mathbb{H}})^* \colon [\mathbb{H}P^3, \mathbb{H}P^2] \to [\mathbb{H}P^2, \mathbb{H}P^2]$$

is trivial. Hence, the Puppe sequence associated to the cofibration $\mathbb{S}^{11} \xrightarrow{\gamma_{2,\mathbb{H}}} \mathbb{H}P^2 \hookrightarrow \mathbb{H}P^3$ leads to the exact sequence

$$\cdots \to [E\mathbb{H}P^2, \mathbb{H}P^2] \xrightarrow{(E\gamma_{2,\mathbb{H}})^*} [\mathbb{S}^{12}, \mathbb{H}P^2] \xrightarrow{(p_{3,\mathbb{H}})^*} [\mathbb{H}P^3, \mathbb{H}P^2] \to 0.$$

Consequently, there is a bijection

$$[\mathbb{S}^{12}, \mathbb{H}P^2]/(E\gamma_{2,\mathbb{H}})^*[E\mathbb{H}P^2, \mathbb{H}P^2] \xrightarrow{\approx} [\mathbb{H}P^3, \mathbb{H}P^2].$$

But, $[E \mathbb{H}P^2, \mathbb{H}P^2] = [E \mathbb{H}P^2, \mathbb{H}P^{\infty}] = [\mathbb{H}P^2, \mathbb{S}^3] = \mathbb{Z}_2\{\nu'\eta_6\eta_7p_{2,\mathbb{H}}\}$. Furthermore, by the proof of Proposition 2.3(2), we have $\nu'\eta_6\eta_7p_{2,\mathbb{H}}\gamma_{2,\mathbb{H}} = 0$. Consequently, $(E\gamma_{2,\mathbb{H}})^*[E\mathbb{H}P^2,\mathbb{H}P^2] = 0$ and we get the required bijection

$$\pi_{11}(\mathbb{S}^3) = \mathbb{Z}_2\{\varepsilon_3\} \approx [\mathbb{S}^{12}, \mathbb{H}P^2] \xrightarrow{(p_{3,\mathbb{H}})^*} [\mathbb{H}P^3, \mathbb{H}P^2].$$
 Q.E.D.

Next, recall that [29, (14.1)–(14.3) Theorems and Remark] state:

Theorem 2.5. (1) If $n \ge 1$ then $\pi^{4n-1}(\mathbb{H}P^n) \approx \mathbb{Z}_2$ is generated by $\eta_{4n-1}p_{n,\mathbb{H}}$.

(2) If $n \geq 2$ then $\pi^{4n-2}(\mathbb{H}P^n) \approx \mathbb{Z}_2$ is generated by $\eta^2_{4n-2}p_{n,\mathbb{H}}$, and

$$\pi^{4n-3}(\mathbb{H}P^n) \approx \begin{cases} 0, & \text{for } n \equiv 0, 2 \pmod{6}; \\ \mathbb{Z}_4, \ \mathbb{Z}_{12} \text{ or } \mathbb{Z}_{24} & \text{for } n \equiv 1 \pmod{6}; \\ \mathbb{Z}_2, \ \mathbb{Z}_4 \text{ or } \mathbb{Z}_8 & \text{for } n \equiv 3, 5 \pmod{6}; \\ \mathbb{Z}_3 & \text{for } n \equiv 4 \pmod{6} \end{cases}$$

is generated by $\nu_{4n-3}^+ p_{n,\mathbb{H}}$.

(3) If $n \ge 3$ then $\pi^{4n-4}(\mathbb{H}P^n) \approx \mathbb{Z}$, and $\pi^{4n-5}(\mathbb{H}P^n) \approx \mathbb{Z}_2$.

(4) If $n \ge 4$ then $\pi^{4n-6}(\mathbb{H}P^n) \approx \mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $\pi^{4n-7}(\mathbb{H}P^n)$ has order at most 5,760.

(5) If $n \geq 5$ then $\pi^{4n-8}(\mathbb{H}P^n) \approx \mathbb{Z} \oplus \mathbb{Z}_2$ or $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

First, to complete Theorem 2.5, we recall that by [13] there is a bijection

$$\pi^{n}(\mathbb{H}P^{2}) = (p_{2,\mathbb{H}})^{*}\pi_{8}(\mathbb{S}^{n}) \approx \pi_{8}(\mathbb{S}^{n})/(\nu_{5}^{+})^{*}\pi_{5}(\mathbb{S}^{n}).$$

Now, we present some details on that result.

 $\textbf{Proposition 2.6. } \pi^{8-k}(\mathbb{H}P^2) \approx \begin{cases} 0 & \text{for } k < 0 \text{ or } k = 3,7,8, \\ \mathbb{Z} & \text{for } k = 0, \\ \mathbb{Z}_2 & \text{for } k = 1,2,4,5,6. \end{cases}$

Proof. Certainly, $\pi^{8-k}(\mathbb{H}P^2) = 0$ if k < 0, $\pi^{8-k}(\mathbb{H}P^2) \approx \mathbb{Z}$ if k = 0 and $\pi^{8-k}(\mathbb{H}P^2) = 0$ if k = 7, 8. Next, consider the exact sequence

$$\cdots \to [E\mathbb{H}P^1, \mathbb{S}^{8-k}] \xrightarrow{(E\gamma_{1,\mathbb{H}})^*} [\mathbb{S}^8, \mathbb{S}^{8-k}] \to [\mathbb{H}P^2, \mathbb{S}^{8-k}] \to [\mathbb{H}P^1, \mathbb{S}^{8-k}] \xrightarrow{(\gamma_{1,\mathbb{H}})^*} [\mathbb{S}^7, \mathbb{S}^{8-k}]$$

associated to the cofibration $\mathbb{S}^7 \xrightarrow{\gamma_{1,\mathbb{H}}} \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^2$ and recall that $\gamma_{1,\mathbb{H}} \equiv \pm(\nu_4 + \alpha_1(4)) \pmod{E\nu'}$ for $\gamma_{1,\mathbb{H}} \colon \mathbb{S}^7 \to \mathbb{H}P^1 = \mathbb{S}^4$.

If k = 1, 2 then we get

$$\pi^{7}(\mathbb{H}P^{2}) = \mathbb{Z}_{2}\{\eta_{7}p_{2,\mathbb{H}}\}$$
 and $\pi^{6}(\mathbb{H}P^{2}) = \mathbb{Z}_{2}\{\eta_{6}^{2}p_{2,\mathbb{H}}\},\$

respectively.

If k = 3 then we easily deduce that

$$\pi^5(\mathbb{H}P^2) = 0.$$

If k = 4 then $[\mathbb{S}^7, \mathbb{S}^4] = \mathbb{Z}\{\nu_4^+\} \oplus \mathbb{Z}_4\{E\nu'\}, [E\mathbb{H}P^1, \mathbb{S}^4] \approx [\mathbb{S}^5, \mathbb{S}^4] = \mathbb{Z}_2\{\eta_4\}, [\mathbb{S}^8, \mathbb{S}^4] = \mathbb{Z}_2\{\nu_4\eta_7\} \oplus \mathbb{Z}_2\{(E\nu')\eta_7\} \text{ and } \eta_4\nu_5 = (E\nu')\eta_7.$ Hence, we derive that

$$\pi^4(\mathbb{H}P^2) = \mathbb{Z}_2\{\nu_4\eta_7 p_{2,\mathbb{H}}\}\$$

If k = 5, since $[\mathbb{S}^7, \mathbb{S}^3] = \mathbb{Z}_2\{\nu'\eta_6\}$, $[\mathbb{S}^4, \mathbb{S}^3] = \mathbb{Z}_2\{\eta_3\}$ and $\eta_3\nu_4 = \nu'\eta_6$, we deduce that the map $(\gamma_{1,\mathbb{H}})^* : [\mathbb{S}^4, \mathbb{S}^3] \to [\mathbb{S}^7, \mathbb{S}^3]$ is an isomorphism. Next, $[\mathbb{S}^8, \mathbb{S}^3] = \mathbb{Z}_2\{\nu'\eta_6^2\}$ and $[E\mathbb{H}P^1, \mathbb{S}^3] \approx [\mathbb{S}^5, \mathbb{S}^3] = \mathbb{Z}_2\{\eta_3^2\}$. Then, the relations $\eta_3\nu_4 = \nu'\eta_6$ and $E(\eta_2\nu') = 0$ imply that the map $(E\gamma_{1,\mathbb{H}})^* : [E\mathbb{H}P^1, \mathbb{S}^3] \to [\mathbb{S}^8, \mathbb{S}^3]$ is trivial. Consequently, we have

$$\pi^{3}(\mathbb{H}P^{2}) = \mathbb{Z}_{2}\{\nu'\eta_{6}^{2}p_{2,\mathbb{H}}\}$$

If k = 6, since $[\mathbb{S}^7, \mathbb{S}^2] = \mathbb{Z}_2\{\eta_2\nu'\eta_6\}$, $[\mathbb{S}^4, \mathbb{S}^2] = \mathbb{Z}_2\{\eta_2^2\}$, and $\eta_3\nu_4 = \nu'\eta_6$, we deduce that the map $(\gamma_{1,\mathbb{H}})^* \colon [\mathbb{S}^4, \mathbb{S}^2] \to [\mathbb{S}^7, \mathbb{S}^2]$ is an isomorphism. Next, $[\mathbb{S}^8, \mathbb{S}^2] = \mathbb{Z}_2\{\eta_2\nu'\eta_6^2\}$ and $[E\mathbb{H}P^1, \mathbb{S}^2] \approx [\mathbb{S}^5, \mathbb{S}^2] = \mathbb{Z}_2\{\eta_2^3\}$. Then, the relations $E(\eta_2\nu') = 0$ and $\eta_3\nu_4 = \nu'\eta_6$ ([28, Lemma 5.7 and (5.9)]) imply that the map $(E\gamma_{1,\mathbb{H}})^* \colon [E\mathbb{H}P^1, \mathbb{S}^2] \to [\mathbb{S}^8, \mathbb{S}^2]$ is trivial. Consequently, we have

$$\pi^2(\mathbb{H}P^2) = \mathbb{Z}_2\{\eta_2\nu'\eta_6^2p_{2,\mathbb{H}}\}$$

and the proof is complete.

Remark 2.7. (1) By making use of the Hopf fibration $\eta_2: \mathbb{S}^3 \to \mathbb{S}^2$, we have the exact sequence

$$0 = [E^k \mathbb{H}P^n, \mathbb{S}^1] \to [E^k \mathbb{H}P^n, \mathbb{S}^3] \xrightarrow{\eta_{2*}} [E^k \mathbb{H}P^n, \mathbb{S}^2] \xrightarrow{i_*} [E^k \mathbb{H}P^n, B\mathbb{S}^1]$$

for $k \ge 0$ and $n \ge 1$. But, the group $[E^k \mathbb{H}P^n, B\mathbb{S}^1] \approx H^2(E^k \mathbb{H}P^n; \mathbb{Z}) = 0$. Consequently, we get the bijection

$$\eta_{2*} \colon \pi^3(E^k \mathbb{H}P^n) \xrightarrow{\approx} \pi^2(E^k \mathbb{H}P^n)$$

Q.E.D.

for $k \geq 0$ and $n \geq 1$. Then, the homotopy equivalence $\mathbb{S}^3 \simeq \Omega(\mathbb{H}P^\infty)$ leads to a bijection

$$\pi^2(E^k \mathbb{H}P^n) \xrightarrow{\approx} [E^{k+1} \mathbb{H}P^n, \mathbb{H}P^{n+[\frac{k+1}{4}]}]$$

for $k \ge 0$ and $n \ge 1$ with the integral part [r] of a real number r.

(2) The cohomotopy groups $\pi^n(E^{n+k}\mathbb{H}P^2)$ for $|k| \leq 3, n \geq 1$ have been described in [12] and for $k = 4, 5, n \geq 2$ in [17]. Furthermore, (1) implies bijections

$$\pi^{2}(E^{k+2}\mathbb{H}P^{2}) \approx \begin{cases} [E^{k+3}\mathbb{H}P^{2}, \mathbb{H}P^{2}] & \text{for } -3 \leq k \leq 0, \\ [E^{k+3}\mathbb{H}P^{2}, \mathbb{H}P^{3}] & \text{for } 1 \leq k \leq 4, \\ [E^{k+3}\mathbb{H}P^{2}, \mathbb{H}P^{4}] & \text{for } k = 5. \end{cases}$$

Now, we update Theorem 2.5(2),(4) and apply Proposition 1.4, to describe $\pi^{4n-k}(E^{l}\mathbb{H}P^{n})$ for $k+l \leq 6$ with $k \geq 0, l \geq 1$ and $n \geq 3$. Setting (a,b) for the greatest common divisor of integers a, b, we state:

Proposition 2.8. If $n \ge 3$ then:

(1)
$$\pi^{4n-3}(\mathbb{H}P^n) \approx \pi_{4n}(\mathbb{S}^{4n-3})/(24, n-1)\pi_{4n}(\mathbb{S}^{4n-3})^1;$$

(2) $\pi^{4n-6}(\mathbb{H}P^n) \approx \begin{cases} \pi^{4n-6}(\mathbb{H}P^{n-1}) = \mathbb{Z}_2\{\eta^2_{4n-2}p_{n-1,\mathbb{H}}\} & \text{for } n \text{ odd,} \\ \pi_{4n}(\mathbb{S}^{4n-6}) \oplus \pi^{4n-6}(\mathbb{H}P^{n-1}) = \mathbb{Z}_2\{\nu^2_{4n-6}\} \oplus \mathbb{Z}_2\{\eta^2_{4n-2}p_{n-1,\mathbb{H}}\} & \text{for } n \text{ even;} \end{cases}$

(3)
$$\pi^{4n-k}(E\mathbb{H}P^n) \approx \begin{cases} \pi_{4n+1}(\mathbb{S}^{4n-1}) & \text{for } k = 0, \\ \pi_{4n+1}(\mathbb{S}^{4n-1}) & \text{for } k = 1, \\ \pi_{4n+1}(\mathbb{S}^{4n-2})/(24, n-1)\pi_{4n+1}(\mathbb{S}^{4n-2}) & \text{for } k = 2, \\ \mathbb{Z} & \text{for } k = 3, \\ \pi_{4n-3}(\mathbb{S}^{4n-4}) & \text{for } k = 4 \end{cases}$$

and there is the short exact sequence

$$0 \to \pi_{4n+1}(\mathbb{S}^{4n-5})/(24, n-1)\pi_{4n+1}(\mathbb{S}^{4n-5}) \to \pi^{4n-5}(E\mathbb{H}P^n) \to \pi_{4n-3}(\mathbb{S}^{4n-5}) \to 0;$$

(4)
$$\pi^{4n-k}(E^2 \mathbb{H}P^n) \approx \begin{cases} \pi_{4n+2}(\mathbb{S}^{4n}) & \text{for } k = 0, \\ \pi_{4n+2}(\mathbb{S}^{4n-1})/(24, n-1)\pi_{4n+2}(\mathbb{S}^{4n-1}) & \text{for } k = 1, \\ \mathbb{Z} & \text{for } k = 2, \\ \pi_{4n-2}(\mathbb{S}^{4n-3}) & \text{for } k = 3 \end{cases}$$

and there is the short exact sequence

$$0 \to \pi_{4n+2}(\mathbb{S}^{4n-4})/(24, n-1)\pi_{4n+2}(\mathbb{S}^{4n-4}) \to \pi^{4n-4}(E^2\mathbb{H}P^n) \to \pi_{4n-2}(\mathbb{S}^{4n-4}) \to 0;$$

(5)
$$\pi^{4n-k}(E^3 \mathbb{H}P^n) \approx \begin{cases} \pi_{4n+3}(\mathbb{S}^{4n})/(24, n-1)\pi_{4n+3}(\mathbb{S}^{4n}) & \text{for } k=0, \\ \mathbb{Z} & \text{for } k=1, \\ \pi_{4n-1}(\mathbb{S}^{4n-2}) & \text{for } k=2 \end{cases}$$

and there is the short exact sequence

$$0 \to \pi_{4n+3}(\mathbb{S}^{4n-3})/(24, n-1)\pi_{4n+3}(\mathbb{S}^{4n-3}) \to \pi^{4n-3}(E^3\mathbb{H}P^n) \to \pi_{4n-1}(\mathbb{S}^{4n-3}) \to 0;$$

 $^{^{1}\}mathrm{We}$ have been informed by J. Mukai that this result has been also shown by H. Sunohara in 2000.

(6)
$$\pi^{4n-k}(E^4 \mathbb{H}P^n) \approx \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ \pi_{4n}(\mathbb{S}^{4n-1}) & \text{for } k = 1 \end{cases}$$

and there is the short exact sequence

$$0 \to \pi_{4n+4}(\mathbb{S}^{4n-2})/(24, n-1)\pi_{4n+4}(\mathbb{S}^{4n-2}) \to \pi^{4n-2}(E^4\mathbb{H}P^n) \to \pi_{4n}(\mathbb{S}^{4n-2}) \to 0;$$

(7) $\pi^{4n}(E^5 \mathbb{H}P^n) \approx \pi_{4n+1}(\mathbb{S}^{4n})$ and there is the short exact sequence

$$0 \to \pi_{4n+5}(\mathbb{S}^{4n-1})/(24, n-1)\pi_{4n+5}(\mathbb{S}^{4n-1}) \to \pi^{4n-1}(E^5\mathbb{H}P^n) \to \pi_{4n+1}(\mathbb{S}^{4n-1}) \to 0;$$

(8) there is the short exact sequence

$$0 \to \pi_{4n+6}(\mathbb{S}^{4n})/(24, n-1)\pi_{4n+6}(\mathbb{S}^{4n}) \to \pi^{4n}(E^6 \mathbb{H}P^n) \to \pi_{4n+2}(\mathbb{S}^{4n}) \to 0.$$

Proof. (1): Since $[\mathbb{H}P^{n-1}, \mathbb{S}^{4n-3}] = 0$, the Puppe sequence associated to the cofibration

$$\mathbb{S}^{4n-1} \xrightarrow{\gamma_{n-1,\mathbb{H}}} \mathbb{H}P^{n-1} \hookrightarrow \mathbb{H}P^n$$

yields the exact sequence

$$\cdots \to [E\mathbb{H}P^n, \mathbb{S}^{4n-3}] \to [E\mathbb{H}P^{n-1}, \mathbb{S}^{4n-3}] \xrightarrow{(E\gamma_{n-1,\mathbb{H}})^*} [\mathbb{S}^{4n}, \mathbb{S}^{4n-3}] \to [\mathbb{H}P^n, \mathbb{S}^{4n-3}] \to 0.$$

Next, the suspension bijection $[\mathbb{H}P^{n-1}, \mathbb{S}^{4n-4}] \xrightarrow{\approx} [E\mathbb{H}P^{n-1}, \mathbb{S}^{4n-3}]$ and $[\mathbb{H}P^{n-1}, \mathbb{S}^{4n-4}] = \mathbb{Z}\{p_{n-1,\mathbb{H}}\}$ lead to $\mathrm{Im}([E\mathbb{H}P^{n-1}, \mathbb{S}^{4n-3}] \xrightarrow{(E\gamma_{n-1,\mathbb{H}})^*} [\mathbb{S}^{4n}, \mathbb{S}^{4n-3}]) = (n-1)[\mathbb{S}^{4n}, \mathbb{S}^{4n-3}] = (24, n-1)[\mathbb{S}^{4n}, \mathbb{S}^{4n-3}].$ Consequently,

$$\pi^{4n-3}(\mathbb{H}P^n) \approx \pi_{4n}(\mathbb{S}^{4n-3})/(24, n-1)\pi_{4n}(\mathbb{S}^{4n-3}).$$

(2): Since $[\mathbb{S}^{4n-1}, \mathbb{S}^{4n-6}] = 0$, the Puppe sequence associated to the cofibration

$$\mathbb{S}^{4n-1} \xrightarrow{\gamma_{n-1},\mathbb{H}} \mathbb{H}P^{n-1} \hookrightarrow \mathbb{H}P^n$$

yields the exact sequence

$$\cdots \to [E\mathbb{H}P^{n-1}, \mathbb{S}^{4n-6}] \xrightarrow{(E\gamma_{n-1,\mathbb{H}})^*} [\mathbb{S}^{4n}, \mathbb{S}^{4n-6}] \to [\mathbb{H}P^n, \mathbb{S}^{4n-6}] \to [\mathbb{H}P^{n-1}, \mathbb{S}^{4n-6}] \to 0.$$

Next, the suspension bijection $[\mathbb{H}P^{n-1}, \mathbb{S}^{4n-7}] \xrightarrow{\approx} [E\mathbb{H}P^{n-1}, \mathbb{S}^{4n-6}]$ and $[\mathbb{H}P^{n-1}, \mathbb{S}^{4n-7}] = \{\nu_{4n-7}^+ p_{n-1,\mathbb{H}}\}$ lead to:

(i): if n is even then the map $[E \mathbb{H} P^{n-1}, \mathbb{S}^{4n-6}] \xrightarrow{(E\gamma_{n-1,\mathbb{H}})^*} [\mathbb{S}^{4n}, \mathbb{S}^{4n-6}]$ is surjective. Consequently,

$$\pi^{4n-6}(\mathbb{H}P^n) \approx \pi^{4n-6}(\mathbb{H}P^{n-1}) = \mathbb{Z}_2\{\eta^2_{4n-6}p_{n-1,\mathbb{H}}\}$$

(ii): if n is odd then the map $[E \mathbb{H} P^{n-1}, \mathbb{S}^{4n-6}] \xrightarrow{(E\gamma_{n-1,\mathbb{H}})^*} [\mathbb{S}^{4n}, \mathbb{S}^{4n-6}]$ is trivial. Hence, we have the short exact sequence

$$0 \to [\mathbb{S}^{4n}, \mathbb{S}^{4n-6}] \to [\mathbb{H}P^n, \mathbb{S}^{4n-6}] \to [\mathbb{H}P^{n-1}, \mathbb{S}^{4n-6}] \to 0.$$

Since $\eta_{4n-2}^2 p_{n-1,\mathbb{H}} \gamma_{n-1,\mathbb{H}} = (n-1)\eta_{4n-2}^2 \nu_{4n-4}^+ = 0$, we get an extension $\overline{\eta_{4n-6}^2 p_{n-1,\mathbb{H}}} = \eta_{4n-6} \overline{\eta_{4n-5}} p_{n-1,\mathbb{H}} \in [\mathbb{H}P^n, \mathbb{S}^{4n-6}]$ of the generator $\eta_{4n-6}^2 p_{n-1,\mathbb{H}} \in [\mathbb{H}P^{n-1}, \mathbb{S}^{4n-6}]$ (Theorem 2.5(2)) which leads to a splitting of that short sequence. Hence,

$$\pi^{4n-6}(\mathbb{H}P^n) \approx \pi_{4n}(\mathbb{S}^{4n-6}) \oplus \pi^{4n-6}(\mathbb{H}P^{n-1}) = \mathbb{Z}_2\{\nu_{4n-6}^2\} \oplus \mathbb{Z}_2\{\eta_{4n-2}^2 p_{n-1,\mathbb{H}}\}.$$

(3)–(8): Let $n \geq 3$. Notice that Proposition 1.3 and the pushout

yield $\mathbb{H}P^n/\mathbb{H}P^{n-2} \simeq \mathbb{S}^{4n-4} \cup_{(n-1)\nu_{4(n-1)}^+} e^{4n}$. Thus, we have a cofibration

$$\mathbb{S}^{4n-1} \xrightarrow{(n-1)\nu_{4(n-1)}^+} \mathbb{S}^{4n-4} \hookrightarrow \mathbb{H}P^n / \mathbb{H}P^{n-2}$$

which leads to the exact sequence

$$\cdots \to [\mathbb{S}^{4n+l-3}, \mathbb{S}^{4n-k}] \xrightarrow{(n-1)(\nu_{4n+l-3}^+)^*} [\mathbb{S}^{4n+l}, \mathbb{S}^{4n-k}] \to [E^l(\mathbb{H}P^n/\mathbb{H}P^{n-2}), \mathbb{S}^{4n-k}] \to [\mathbb{S}^{4n+l-4}, \mathbb{S}^{4n-k}] \xrightarrow{(n-1)(\nu_{4n+l-4}^+)^*} [\mathbb{S}^{4n+l-1}, \mathbb{S}^{4n-k}]$$

for $k, l \ge 0$ and $n \ge 3$. Next, in view of Proposition 1.4, we have

$$[E^{l}\mathbb{H}P^{n},\mathbb{S}^{4n-k}]\approx [E^{l}(\mathbb{H}P^{n}/\mathbb{H}P^{n-2}),\mathbb{S}^{4n-k}]$$

for $k + l \leq 6$ with $k, l \geq 0$ and $n \geq 3$.

Then, we apply presentations of homotopy groups $\pi_{n+i}(\mathbb{S}^n)$ for $0 \le i \le 6$ ([28, Chapter V]) and $\nu_n^+\eta_{n+5} = \nu_n\eta_{n+5} = 0$ for $n \ge 6$ ([28, (3.9)]). If l = 0 then we get $\pi^{4n}(\mathbb{H}P^n) \approx \mathbb{Z}$ and $\pi^{4n-k}(\mathbb{H}P^n)$ for $1 \le k \le 6$ as stated in Theorem 2.5. Whereas, for $1 \le l \le 6$ we get the statements (1)–(6) and the proof is complete.

Since $E^k \mathbb{H}P^m$ is 1-connected, we have $[E^k \mathbb{H}P^m, \mathbb{R}P^n] = [E^k \mathbb{H}P^m, \mathbb{S}^n]$ for $k \geq 0$. Next, $H^2(E^k \mathbb{H}P^m; \mathbb{Z}) = 0$ and Lemma 1.5 yields $[E^k \mathbb{H}P^m, \mathbb{C}P^n] = [E^k \mathbb{H}P^m, \mathbb{S}^{2n+1}]$ for $k \geq 0$. Now, consider 4m < n and notice that $[\mathbb{H}P^m, \mathbb{S}^n] = [\mathbb{H}P^m, \mathbb{S}^\infty] = 0$. This implies $[\mathbb{H}P^m, \mathbb{R}P^n] =$ $[\mathbb{H}P^m, \mathbb{R}P^\infty] = 0$. Also, by Hopf Theorem $[\mathbb{H}P^m, \mathbb{R}P^{4m}] = [\mathbb{H}P^m, \mathbb{S}^{4m}] = H^{4m}(\mathbb{H}P^m; \mathbb{Z}) \approx \mathbb{Z}$. Consequently, applying Theorem 2.5 and Proposition 2.8, we may state:

Corollary 2.9. (1) If $n \ge 1$ then $[\mathbb{H}P^n, \mathbb{C}P^{2n-1}] \approx [\mathbb{H}P^n, \mathbb{R}P^{4n-1}] \approx \mathbb{Z}_2;$

- (2) if $n \geq 2$ then $[\mathbb{H}P^n, \mathbb{R}P^{4n-2}] \approx \mathbb{Z}_2;$
- (3) if $n \ge 2$ then $[\mathbb{H}P^n, \mathbb{C}P^{2n-2}] \approx [\mathbb{H}P^n, \mathbb{R}P^{4n-3}] \approx \frac{[\mathbb{S}^{4n}, \mathbb{S}^{4n-3}]}{(n-1)[\mathbb{S}^{4n}, \mathbb{S}^{4n-3}]};$
- (4) if $n \geq 3$ then $[\mathbb{H}P^n, \mathbb{R}P^{4n-4}] \approx \mathbb{Z};$

- (5) if $n \ge 2$ then $[\mathbb{H}P^n, \mathbb{C}P^{2n-3}] \approx [\mathbb{H}P^n, \mathbb{R}P^{4n-5}] \approx \mathbb{Z}_2;$
- (6) if $n \ge 4$ then $[\mathbb{H}P^n, \mathbb{R}P^{4n-6}] \approx \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is odd,} \\ \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } n \text{ is even;} \end{cases}$
- (7) if $n \ge 4$ then $[\mathbb{H}P^n, \mathbb{C}P^{2n-4}] \approx [\mathbb{H}P^n, \mathbb{R}P^{4n-7}]$ has order at most 5,760;
- (8) if $n \ge 5$ then $[\mathbb{H}P^n, \mathbb{R}P^{4n-8}] \approx \mathbb{Z} \oplus \mathbb{Z}_2$ or $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Applying Theorems 1.2, 2.5 and Proposition 2.8, we get:

Corollary 2.10. (1) If $n \ge 1$ then $[E \mathbb{H} P^n, \mathbb{R} P^{4n}] \approx \mathbb{Z}_2$;

- (2) if $n \ge 2$ then $[E \mathbb{H} P^n, \mathbb{R} P^{4n-1}] \approx [E \mathbb{H} P^n, \mathbb{C} P^{2n-1}] \approx \mathbb{Z}_2;$
- (3) if $n \ge 2$ then $[E\mathbb{H}P^n, \mathbb{R}P^{4n-2}] \approx \frac{[\mathbb{S}^{4n}, \mathbb{S}^{4n-3}]}{(n-1)[\mathbb{S}^{4n}, \mathbb{S}^{4n-3}]};$
- (4) if $n \ge 3$ then $[E \mathbb{H} P^n, \mathbb{C} P^{2n-2}] \approx [E \mathbb{H} P^n, \mathbb{R} P^{4n-3}] \approx \mathbb{Z};$
- (5) if $n \ge 3$ then $[E \mathbb{H} P^n, \mathbb{R} P^{4n-4}] \approx \mathbb{Z}_2;$
- (6) if $n \ge 4$ then $[E \mathbb{H} P^n, \mathbb{C} P^{2n-3}] \approx [E \mathbb{H} P^n, \mathbb{R} P^{4n-5}] \approx \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is odd;} \\ \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } n \text{ is even;} \end{cases}$
- (7) if $n \ge 4$ then $[E \mathbb{H} P^n, \mathbb{R} P^{4n-6}]$ has order at most 5,760;
- (8) if $n \ge 5$ then $[E \mathbb{H} P^n, \mathbb{C} P^{2n-4}] \approx [E \mathbb{H} P^n, \mathbb{R} P^{4n-7}] \approx \mathbb{Z} \oplus \mathbb{Z}_2$ or $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We close the paper with:

- **Remark 2.11.** (1) Lemma 1.5 leads to $[E^k \mathbb{H}P^m, \mathbb{C}P^n] \approx [E^k \mathbb{H}P^m, \mathbb{S}^{2n+1}]$ for $k, m, n \geq 0$. Hence, for k + 4m < 2n + 1 we have $[E^k \mathbb{H}P^m, \mathbb{C}P^n] \approx [E^k \mathbb{H}P^m, \mathbb{S}^{2n+1}] = 0$ and $[E^{2n-4m+1} \mathbb{H}P^m, \mathbb{C}P^n] \approx [E^{2n-4m+1} \mathbb{H}P^m, \mathbb{S}^{2n+1}] \approx \mathbb{Z}$.
 - If $m \leq n$ then (by means of skeleton reasons):
 - 2. $[\mathbb{R}P^{4m+k}, \mathbb{H}P^n] \approx [\mathbb{R}P^{4m+k}, \mathbb{H}P^m] \approx [\mathbb{R}P^{4m+k}, \mathbb{H}P^\infty]$ for k = 0, 1, 2;
 - 3. $[\mathbb{C}P^{2m+k}, \mathbb{H}P^n] \approx [\mathbb{C}P^{2m+k}, \mathbb{H}P^m] \approx [\mathbb{C}P^{2m+k}, \mathbb{H}P^\infty]$ for k = 0, 1;
 - 4. $[\mathbb{H}P^m, \mathbb{H}P^n] \approx [\mathbb{H}P^m, \mathbb{H}P^\infty];$
 - 5. $[E\mathbb{R}P^{4m+k}, \mathbb{H}P^n] \approx [E\mathbb{R}P^{4m+k}, \mathbb{H}P^m] \approx [\mathbb{R}P^{4m+k}, \mathbb{S}^3]$ for k = 0, 1;
 - 6. $[E\mathbb{C}P^{2m+k}, \mathbb{H}P^n] \approx [E\mathbb{C}P^{2m+k}, \mathbb{H}P^m] \approx [\mathbb{C}P^{2m+k}, \mathbb{S}^3]$ for k = 0, 1;
 - 7. $[E \mathbb{H} P^m, \mathbb{H} P^n] \approx [\mathbb{H} P^m, \mathbb{S}^3];$
 - 8. $[E^2 \mathbb{F}P^{\frac{4m}{d}}, \mathbb{H}P^n] \approx [E \mathbb{F}P^{\frac{4m}{d}}, \mathbb{S}^3]$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , where $d = \dim_{\mathbb{R}} \mathbb{F}$.

Acknowledgements

This work was initiated and completed during the first author's visits to Instituto de Geociências e Ciências Exatas, UNESP–Univ Estadual Paulista, Rio Claro–SP, Brazil. He would like to thank the Instituto de Geociências e Ciências Exatas for its hospitality. Special thanks are due to the CAPES–Ciência sem Fronteiras, Processo: 88881.068125/2014-01 for supporting these visits.

The authors greatly appreciate the anonymous referee for careful reading of the manuscript last version and his/her comments and suggestions.

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